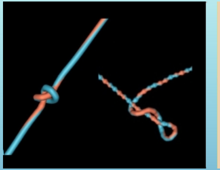
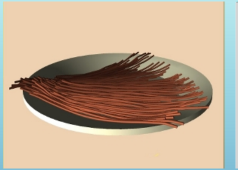
A Nonsmooth Newton Solver for Capturing Exact Coulomb Friction in Fiber Assemblies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Motivation

- Fibers assemblies are common in the real world
- But not much studied in the past
- Contact and dry friction play a major role w.r.t. shape and motion

  (volume, stable stacking, nonsmooth patterns, nonsmooth dynamics)

Fibers assemblies: Previous work

Main motivation

Hair simulation in Computer Graphics

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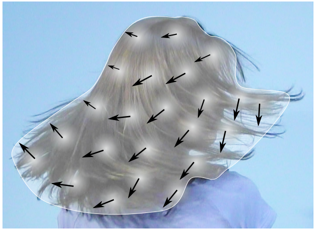
Three families of models

Fibers assemblies: Previous work



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Hair simulation in Computer Graphics

Three families of models

1. Continuum-based [Hadap and Magnenat-Thalmann 2001]
   → Hair medium governed by fluid-like equations

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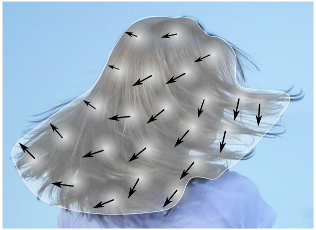

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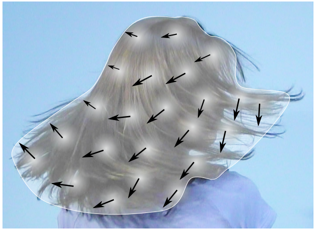

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Hair simulation in Computer Graphics

Three families of models

1. Continuum-based [Hadap and Magnenat-Thalmann 2001]
   → Hair medium governed by fluid-like equations
   - ☺ Macroscopic, intrinsic interaction model
   - ☹ No discontinuities

Fibers assemblies: Previous work

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Hair simulation in Computer Graphics

Three families of models

2. Wisp-based (or fiber-based) [Plante et al. 2001]
   $\rightarrow$ A set of strands primitives combined with a simple interaction model

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   ☺ Allows for fine-grain simulations [Selle et al. 2008]

Fibers assemblies: Previous work

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Hair simulation in Computer Graphics

Three families of models

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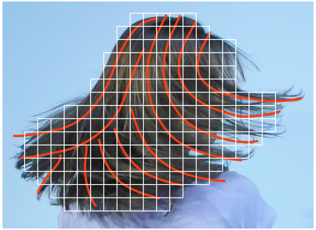
   → A set of strands primitives combined with a simple interaction model

   - ☺ Allows for fine-grain simulations [Selle et al. 2008]
   - ☹ Lack of stability if penalties used
   - ☹ Many contacts omitted → lack of volume
   - ☹ No dry friction (viscous model)



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Hair simulation in Computer Graphics

Three families of models

3. Mixed of the two others [Mc Adams et al. 2009]
   → A mixed Eulerian-Lagrangian contact formulation

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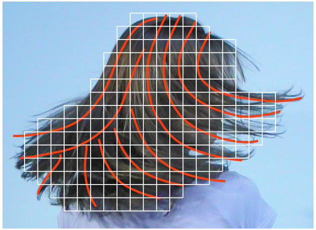
Fibers assemblies: Previous work



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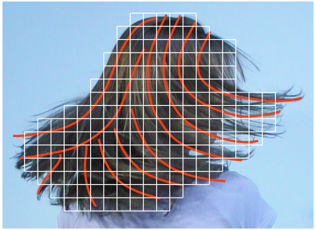

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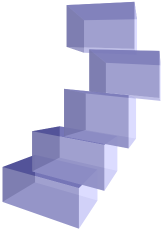
Hair simulation in Computer Graphics

Three families of models

3. Mixed of the two others [Mc Adams et al. 2009]
   $\rightarrow$ A mixed Eulerian-Lagrangian contact formulation
   - Global volume preservation together with detailed features
   - Still no dry friction

Frictional contact in Computer Graphics

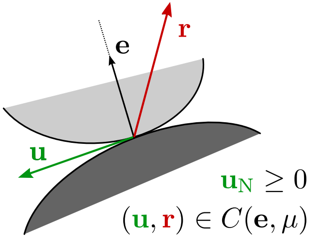
In contrast, dry friction has been considered for a long time in Computer Graphics for the simulation of rigid bodies



Frictional contact: Previous work

Ideal model for frictional contact

Non-penetration + Coulomb friction



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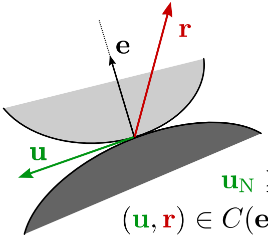


$$\mathbf{u}_{\mathrm{N}} \geq 0$$

$$(\mathbf{u}, \mathbf{r}) \in C(\mathbf{e}, \mu)$$

Most robust approach

Implicit constrained-based [Baraff 1994, Erleben 2007, Kaufman et al. 2008, Otaduy et al. 2009]

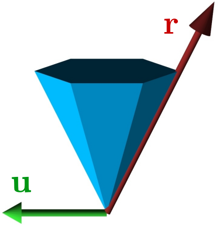
$\rightarrow$ Global formulation where velocities and contact forces are unknown

Implicit constrained-based methods, in practice

Common approximation in Computer Graphics

Linearization of the Coulomb friction cone

$\rightarrow$ Formulation of a Linear Complementarity Problem (LCP)



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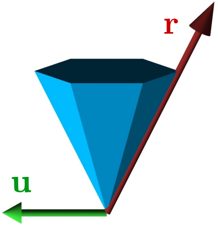

- A bunch of solvers available

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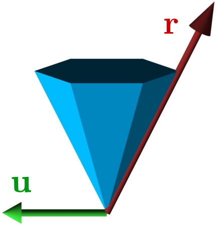

Common approximation in Computer Graphics

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$\rightarrow$ Formulation of a Linear Complementarity Problem (LCP)

- A bunch of solvers available
- Important drift when using too few facets
- Increasing the number of facets results in an explosion of variables

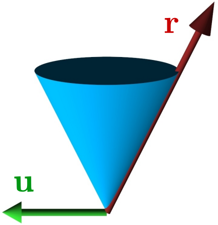
Implicit constrained-based methods, in practice

In contrast...

Implicit constrained-based methods, in practice

In Computational Mechanics

Exact Coulomb law numerically tackled for decades



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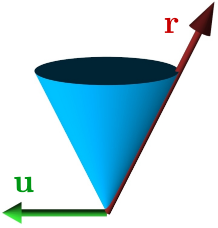


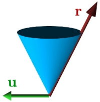
- Main application: simulation of granulars [Moreau 1994, Jean 1999]
- A well-known, exact approach: the [Alart and Curnier 1991] functional formulation

Contributions

- Design a generic Newton algorithm for exact Coulomb friction in fiber assemblies, relying on the Alart and Curnier functional formulation
- Identify a simple criterion for convergence: no over-constraining

Outline

Formulating Contact in Fiber Assemblies



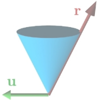
A Newton Algorithm for Exact Coulomb Friction

Results and Convergence Analysis

Discussion and Future Work

Outline

Formulating Contact in Fiber Assemblies



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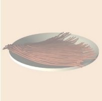

Results and Convergence Analysis

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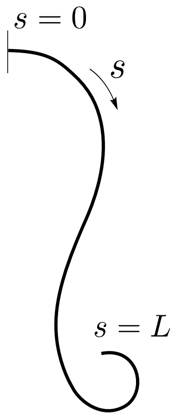
Fiber model

Kirchhoff model for thin elastic rods

- Inextensible
- Elastic bending and twist



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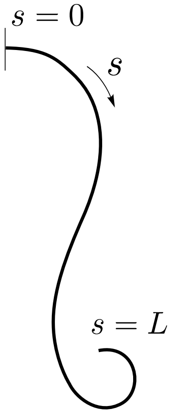
In practice, three rod models used

- Implicit mass-spring system [Baraff et al. 1998]
- CORDE model [Spillmann et al. 2007]
- Super-helices [Bertails et al. 2006]



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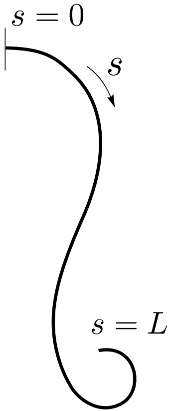
$\rightarrow$ We define a generic discrete rod model:

$$M v + f = 0 \quad \text{and} \quad u = H v + w$$



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Fiber assembly: One-step problem

- Global system (with frictional contact):

$$\begin{cases} \mathbf{M}\,\mathbf{v} + \mathbf{f} & = & \mathbf{H}^\top \mathbf{r} \\ \mathbf{u} & = & \mathbf{H}\,\mathbf{v} + \mathbf{w} \\ (\mathbf{u}, \mathbf{r}) & & \text{satisfies the Coulomb's law} \end{cases} \tag{1}$$

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- Compact formulation in $(\mathbf{u}, \mathbf{r})$:

$$\begin{cases} \mathbf{u} & = & \mathbf{W}\,\mathbf{r} + \mathbf{q} \\ (\mathbf{u}, \mathbf{r}) & & \text{satisfies the Coulomb's law} \end{cases} \tag{2}$$

where $\mathbf{W} = \mathbf{H}\,\mathbf{M}^{-1}\,\mathbf{H}^\top$ is the Delassus operator

Outline

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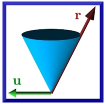

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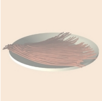

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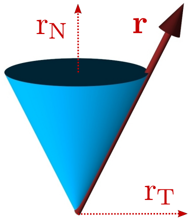
Coulomb's law: disjonctive formulation

Let $\mu \geq 0$ be the friction coefficient.
We define the second-order cone K_μ,

$$K_\mu = \{\|r_{\mathrm{T}}\| \leq \mu r_{\mathrm{N}}\} \subset \mathbb{R}^3$$



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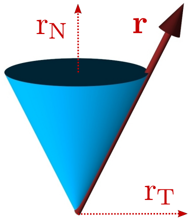

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Frictional contact with Coulomb's law ($\approx$ 1780)

$$(\mathbf{u}, \mathbf{r}) \in C(\mathbf{e}, \mu) \iff$$

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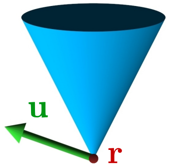

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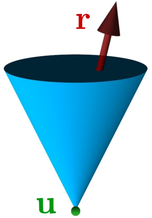

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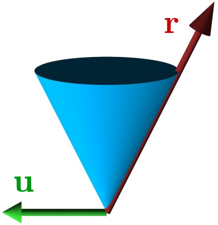

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Coulomb's law: functional formulation

Idea

Express Coulomb's law as $f(u, r) = 0$ with f a nonsmooth function

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Alart and Curnier formulation (1991)

$$\boldsymbol{f}^{AC}(\mathbf{u}, \mathbf{r}) = \begin{bmatrix} f_N^{AC}(\mathbf{u}, \mathbf{r}) \\ \boldsymbol{f}_T^{AC}(\mathbf{u}, \mathbf{r}) \end{bmatrix} = \begin{bmatrix} \mathrm{P}_{\mathbb{R}^+}(r_N - \rho_N u_N) & - & r_N \\ \mathrm{P}_{\boldsymbol{B}(\mathbf{0}, \mu r_N)}(\mathbf{r}_T - \rho_T \mathbf{u}_T) & - & \mathbf{r}_T \end{bmatrix}$$

where $\rho_N, \rho_T \in \mathbb{R}_+^*$ and P_K is the projection onto the convex K.

$$\boxed{(\mathbf{u}, \mathbf{r}) \in C(\mathbf{e}, \mu) \iff \boldsymbol{f}^{AC}(\mathbf{u}, \mathbf{r}) = \mathbf{0}}$$

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Nonsmooth Newton on the Alart-Curnier function

Formulation of the one-step problem

$$\begin{cases} \mathbf{u} & = \quad \mathbf{W}\,\mathbf{r} + \mathbf{q} \\ \boldsymbol{f}^{AC}(\mathbf{u}, \mathbf{r}) & = \quad \mathbf{0} \end{cases}$$

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Solving method: (damped) Newton algorithm

- We minimize $\|\Phi(\mathbf{r})\|^2$
- Requires the computation of $\nabla\Phi$ (subgradients)
- Natural stopping criterion: $\frac{1}{2}\,\|\Phi(\mathbf{r})\|^2 < \epsilon$

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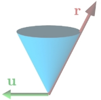

A Newton Algorithm for Exact Coulomb Friction

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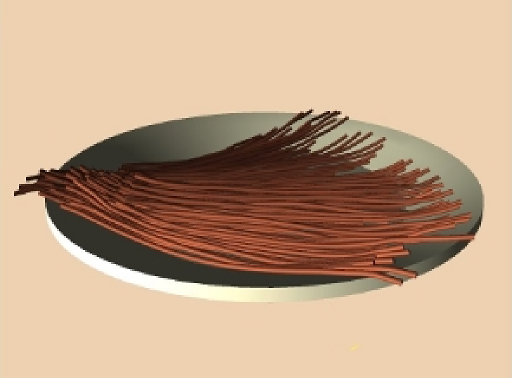
Results



Convergence issues

In theory...

- No proof of existence of a solution to the one-step problem
- No proof of convergence (nonsmooth function)

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In practice

- Our fiber problems are likely to possess a solution [Cadoux 2009]
- We found an empiric criterion for convergence

Convergence analysis

- Let us define $\nu = \frac{3\, n_{\mathrm{contacts}}}{n_{\mathrm{dofs}}}$

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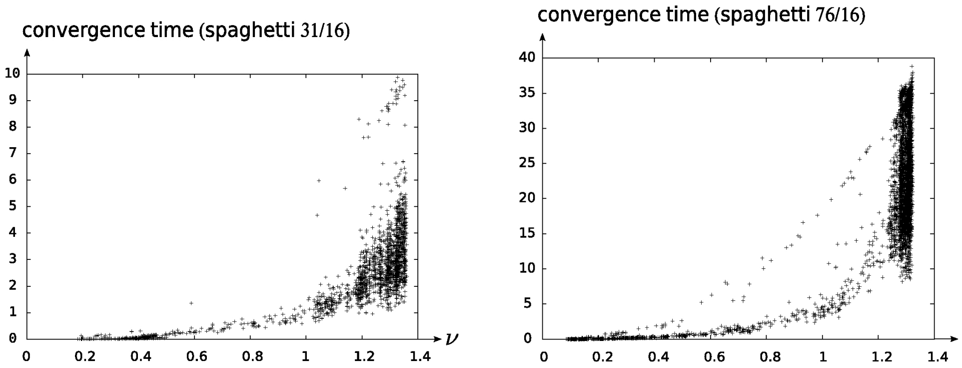
Convergence illustration



Convergence time (in seconds) function of ν

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$\rightarrow$ ν plays the role of a conditioning number for our problem
$\rightarrow$ better suited for assemblies of compliant models than rigid bodies
$\rightarrow$ for over-constrained systems, a splitting strategy seems more appropriate

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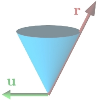
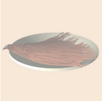


A Newton Algorithm for Exact Coulomb Friction



Results and Convergence Analysis

Discussion and Future Work

SIGGRAPH2011

Conclusions

Contributions

- A generic Newton solver for capturing exact Coulomb friction in fibers

  Relying on the Alart and Curnier functional formulation

- A simple criterion for convergence

  Based on the degree of constraining of the system

SIGGRAPH2011

Conclusions

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Source code

The source code for our solver is freely available on

http://www.inrialpes.fr/bipop/people/bertails/Papiers/nonsmoothNewtonSolverTOG2011.html

Limitations and Future work

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- Does not scale up well (tens to hundreds fibers *vs.* thousands fibers)

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Future work

- Design a robust solver for thousands densely packed rods
- Carefully validate the (hair) collective behavior against real experiments
- Build a macroscopic model for fibrous media (nonsmooth laws)

Follow-up

- An improved functional formulation for exact Coulomb friction
- A splitting algorithm dedicated to large hair problems
  $\rightarrow$ In practice, this modified solver works very well for complex scenarios

The End

Acknowledgments

We are grateful to the anonymous reviewers for their helpful comments.

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Thank You for your attention !